J. DIFFERENTIAL GEOMETRY
14 (1979) 241-254

POSITIVE RICCI CURVATURE ON FIBRE BUNDLES

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In this paper we construct complete metrics of positive Ricci curvature on a large class of fibre bundles. Some of the results for compact fibres have been obtained independently by Poor [12]. The base manifold M is assumed to be compact admitting a metric with $\mathrm{Ric}_M > 0$. If $F = G/H$ is compact homogeneous with $\pi_1(F)$ finite, we show that any bundle over M with fibre F admits a metric with $\mathrm{Ric} > 0$. Certain exotic 7- and 15-spheres arise as sphere bundles over spheres and, thus, admit metrics of positive Ricci curvature.

For vector bundles we have the following result.

Theorem. *Let $\hat{\pi}$: $B \to M$ be a vector bundle over M, a compact manifold admitting a metric of positive Ricci curvature. If the fibre dimension is greater than two, B admits a complete metric of positive Ricci curvature.*

This result is related to a question of Cheeger and Gromoll [1]: Does any vector bundle over S^n admit a complete metric with $K \geqslant 0$? Rigas has some partial results on this problem [13].

The author would like to express his thanks to Professors Shing-Tung Yau, Hans Samelson, and Robert Osserman for helpful conversations throughout the course of this work.

1. Preliminaries

We begin by recalling some basic notions and introducing notation. All objects (manifolds, maps, actions, etc.) will be C^∞, and M^n denotes a manifold of dimension n. The differential of a map f: $M \to N$ between manifolds will usually be abbreviated to $f_p(X)$ of just $f(X)$ for $X \in T_pM$. For a Riemannian manifold M we use the following curvature convention:

$$R_M(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$R_M(X, Y, Z, W) = \langle R_M(X, Y)Z, W\rangle,$$
$$R_M(X, Y) = R_M(X, Y, Y, X).$$

$\mathfrak{X}(M)$ denotes the C^∞ vector fields on M.

Communicated by I. M. Singer, April 4, 1977.

A Lie group G^m acting on M^n from the right induces a Lie algebra homomorphism $\alpha\colon \mathfrak{g} \to \mathfrak{X}(M)$, where $\mathfrak{g}$ is the Lie algebra of G, defined by $\alpha(X)(p) = \overline{X}(p) = c'(0)$, $c(t) = p \cdot \exp(tX)$, $p \in M$. This map can also be expressed as follows: For $p \in M$, define $i_p\colon G \to M$, $i_p(g) = p \cdot g$. Then $\overline{X}(p) = (i_p)_e(X)$, $e =$ identity of G. If the action is free, α is injective. In fact, in that case $\alpha_p\colon \mathfrak{g} \to T_pM$, $\alpha_p(X) = \overline{X}(p)$ is injective.

Suppose further that G has a bi-invariant metric, and M is Riemannian. For each $p \in M$ there exist orthonormal bases $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ of $\mathfrak{g}$ and T_pM respectively and constants $\lambda_1(p), \ldots, \lambda_m(p)$ satisfying $\lambda_1(p) \geqslant \lambda_2(p) \geqslant \cdots \geqslant \lambda_s(p) > \lambda_{s+1}(p) = \cdots = \lambda_m(p) = 0$ such that

$$\begin{aligned} \alpha_p(e_i) &= \lambda_i(p) f_i, \qquad i = 1, \ldots, s, \\ &= 0, \qquad i = s+1, \ldots, m. \end{aligned}$$

The $\lambda_i(p)$ are the expansion factors of α_p and depend continuously on p. In general the bases and the integer s also depend upon p.

A differentiable fibre bundle $(E, M, F, G, \hat{\pi})$ will usually be abbreviated to $\hat{\pi}\colon E \to M$. Here F is the fibre, and G the structural group. Unless stated otherwise we assume throughout that G is a compact Lie group acting on F from the left. The principal bundle $\pi\colon P \to M$ associated to a fibre bundle has a free right action by G such that $P/G = M$. G acts from the right on $P \times F$ by $(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f)$ and $(P \times F)/G = E$.

We now summarize material on Riemannian submersions developed by O'Neill [10]. For Riemannian manifolds M and N consider a surjective map $\pi\colon N \to M$ with π_* of maximal rank at all points. This implies that for $x \in M$, $\pi^{-1}(x)$ is a submanifold of N. For $p \in \pi^{-1}(x)$, denote the tangent space of $\pi^{-1}(x)$ at p by V_p and its orthogonal complement in T_pN by H_p. Such a map π is a Riemannian submersion if $\pi_*|H_p$ is an isometry for all $p \in N$.

There are two (1, 2) tensors on N associated with a Riemannian submersion:

$$\begin{aligned} T_X Y &= (\nabla_{X_v} Y_v)_h + (\nabla_{X_v} Y_h)_v, \\ A_X Y &= (\nabla_{X_h} Y_v)_h + (\nabla_{X_h} Y_h)_v, \end{aligned} \tag{1.1}$$

for $X, Y \in \mathfrak{X}(N)$ where $X = X_h + X_v$ is the decomposition into horizontal and vertical components. The tensors have the following properties:

(1.2)
(i) T_X and A_X are skew-symmetric for $X \in \mathfrak{X}(N)$,
(ii) $T_V W = T_W V$ for vertical vector fields V, W,
(iii) $A_X Y = -A_Y X = \frac{1}{2}[X, Y]_v$ for horizontal vector fields X, Y.

$T = 0$ if and only if the fibres $\pi^{-1}(x)$ are totally geodesic. O'Neill has developed formulas relating the curvatures of M, N, and $\pi^{-1}(x)$. For our applications $T \equiv 0$. We state his results for that case. Set $F_x = \pi^{-1}(x)$.

Theorem 1.1. *Let $\pi: N \to M$ be a Riemannian submersion such that the fibres are totally geodesic ($T \equiv 0$). If $x \in M$, $p \in F_x$, X, $Y \in H_p$, and V, $W \in V_p$, then the following hold:*

$$
\begin{aligned}
&\text{(i)} && R_N(X, Y) = R_M(\pi(X), \pi(Y)) - 3\|A_X Y\|^2,\\
&\text{(ii)} && R_N(V, W) = R_{F_x}(V, W),\\
(1.3)\quad &\text{(iii)} && R_N(X, V) = \|A_X V\|^2,\\
&\text{(iv)} && R_N(V, W, W, X) = 0,\\
&\text{(v)} && R_N(X, Y, Y, V) = \langle (\nabla_Y A)_Y X, V\rangle.
\end{aligned}
$$

2. Curvature on principal bundles

In this section $\pi: P \to M$ denotes a principal bundle over a compact manifold M^n with fibre and group G^m. Fix $\langle\ ,\ \rangle_G$, a bi-invariant metric on G. For a fixed connection ω on P and metric $\langle\ ,\ \rangle_M$ on M there is a family of metrics $\langle\ ,\ \rangle_t$ on P, $t > 0$, defined by

$$\langle X, Y\rangle_t = \langle \pi(X), \pi(Y)\rangle_M + t^2\langle \omega(X), \omega(Y)\rangle_G.$$

These metrics have also been used by Jensen [7], and Lawson and Yau [9]. The map $\pi: P \to M$ becomes a Riemannian submersion for $\langle\ ,\ \rangle_t$, and the vertical and horizontal subspaces of ω agree with the respective vertical and horizontal subspaces of the Riemannian submersion. For X, $Y \in \mathfrak{g}$, $\overline{X}$, $\overline{Y}$ are vertical vector fields and $\langle \overline{X}, \overline{Y}\rangle_t = t^2\langle X, Y\rangle_G$. Since the vertical and horizontal subspaces are invariant under G, the metrics $\langle\ ,\ \rangle_t$ are G-invariant.

Lemma 2.1. *The fibres in P are totally geodesic with respect to $\langle\ ,\ \rangle_t$.*

Proof. It suffices to show $\langle \nabla_{\overline{V}}\overline{V}, H\rangle_t \equiv 0$ for $V \in \mathfrak{g}$ and H an invariant horizontal field. In this case $\langle \overline{V}, H\rangle_t \equiv 0$ and $[\overline{V}, H] = 0$, so $\langle \nabla_{\overline{V}}\overline{V}, H\rangle_t = -\langle \overline{V}, \nabla_{\overline{V}}H\rangle_t = -\langle \overline{V}, \nabla_H \overline{V}\rangle_t = -\frac{1}{2}H\langle \overline{V}, \overline{V}\rangle_t = 0$.

We will denote the various quantities associated to $\langle\ ,\ \rangle_t$ with a subscript or superscript, e.g., ∇^t, R_t, A^t. For $t = 1$ the t will usually be deleted.

Lemma 2.2. *Let X and Y be horizontal fields, and V and W vertical fields on P. The following equations then hold:*

$$
\begin{aligned}
(2.1)\qquad & A^t_X Y = A_X Y, \quad A^t_X V = t^2 A_X V, \quad \nabla^t_X Y = \nabla_X Y,\\
& (\nabla^t_X V)_v = (\nabla_X V)_v = [X, V]_v, \quad (\nabla^t_V W)_v = \nabla_V W.
\end{aligned}
$$

The proof uses only the basic properties of A^t in (1.2), $T \equiv 0$, as well as the standard formulas for the Levi-Civita connection.

The following result combines Theorem 1.1 and Lemma 2.2.

Proposition 2.3. For $p \in P$ let $X, Y \in H_p$ and $V, W \in V_p$. The following equations then hold:

$$
(2.2)\quad
\begin{aligned}
&\text{(i)} && R_t(X, Y) = R_M(\pi(X), \pi(Y)) - 3t^2\|A_X Y\|^2,\\
&\text{(ii)} && R_t(V, W) = t^2 R(V, W) = (t^2/4)\|[\alpha_p^{-1}(V), \alpha_p^{-1}(W)]\|_G^2,\\
&\text{(iii)} && R_t(X, V) = t^4\|A_X V\|^2,\\
&\text{(iv)} && R_t(V, W, W, X) = 0,\\
&\text{(v)} && R_t(X, Y, Y, V) = t^2 R(X, Y, Y, V).
\end{aligned}
$$

Proof. (i), (iii), and (iv) follow easily. Since the fibres are totally geodesic and $R_G(X, Y, Y, X) = (1/4)\|[X, Y]\|_G^2$ for $X, Y \in \mathfrak{g}$, (ii) holds. To prove (v), first extend X, Y and U, V respectively to horizontal and vertical fields which we denote by the same letters. Combining Theorem 1.1, Lemma 2.2 and (1.2) we have

$$
\begin{aligned}
R_t(X, Y, Y, V) &= \langle(\nabla_Y^t A^t)_Y X, V\rangle_t\\
&= \langle\nabla_Y^t(A_Y X) - A^t_{\nabla_Y Y}X - A_Y^t(\nabla_Y X), V\rangle_t\\
&= \langle\nabla_Y(A_Y X) - A_{\nabla_Y Y}X - A_Y(\nabla_Y X), V\rangle_t\\
&= t^2\langle(\nabla_Y A)_Y X, V\rangle = t^2 R(X, Y, Y, V).
\end{aligned}
$$

The following theorem establishes a large class of principal bundles which admit metrics of positive Ricci curvature.

Theorem 2.4. *Let $\pi: P \to M^n$ be a principal bundle with compact semi-simple structural group G^m. If M is compact admitting a metric $\langle\ ,\ \rangle_M$ with $\mathrm{Ric}_M \geqslant M_0 > 0$, then P admits a G-invariant metric of positive Ricci curvature.*

Proof. Fix a connection ω on P and a bi-invariant metric $\langle\ ,\ \rangle_G$ on G. We show that for t small enough $\mathrm{Ric}_t > 0$. Because G is compact semi-simple, $\mathrm{Ric}_G(e_1) = (1/4)\Sigma_i\|[e_1, e_i]\|_G^2$ is positive for $e_1, \ldots, e_m$ an orthonormal basis of $\mathfrak{g}$. Thus there exists a constant G_0 such that $\mathrm{Ric}_G \geqslant G_0 > 0$.

For $X \in T_pP$, $\|X\|_t = 1$, there exist orthonormal bases $e_1, \ldots, e_m$ and $h_1, \ldots, h_n$ of $\mathfrak{g}$ and H_p respectively such that $X = ae_1/t + bh_1$ for some a, b satisfying $a^2 + b^2 = 1$. Here we have identified e_1 with $\bar{e}_1(p)$. With this identification $e_1/t, \ldots, e_m/t, h_1, \ldots, h_n$ is an orthonormal basis of T_pP with respect to $\langle\ ,\ \rangle_t$. Thus $X, be_1/t - ah_1, e_2/t, \ldots, e_m/t, h_2, \ldots, h_n$ is also such a basis. Hence

$$
(2.3)\quad \mathrm{Ric}_t(X) = R_t\Big(X, \frac{b}{t}e_1 - ah_1\Big) + \sum_{i=2}^{m} R_t\Big(X, \frac{e_i}{t}\Big) + \sum_{j=2}^{n} R_t(X, h_j).
$$

Proposition 2.3 and the symmetrics of the curvature tensor imply

$$
\begin{aligned}
& R_t(X, be_1/t - ah_1) = t^2\|A_{h_1}e_1\|^2 \geqslant 0, \\
& R_t(X, e_i/t) \geqslant a^2\|[e_1, e_i]\|_G^2/(4t^2), \\
(2.4) \qquad & R_t(X, h_j) \geqslant b^2\big[R_M(\pi(h_1), \pi(h_j))\big] - 3b^2t^2\|A_{h_1}h_j\|^2 \\
& \qquad + 2abt\, R(e_1, h_j, h_j, h_1).
\end{aligned}
$$

The terms $R(e_1, h_j, h_j, h_1)$ and $\|A_{h_1}h_j\|^2$ have bounds independent of e_i and h_j. Thus by (2.3) and (2.4)

$$
\begin{aligned}
\mathrm{Ric}_t(X) &= b^2 \,\mathrm{Ric}_M(\pi(h_1)) + (a^2/t^2)\,\mathrm{Ric}_G(e_1) + O(t) \\
&\geqslant b^2M_0 + (a^2/t^2)G_0 + O(t).
\end{aligned}
$$

Hence for t small enough, $\mathrm{Ric}_t > 0$.

Remark. The validity of the above theorem requires some assumption on the fibre. For example, $\pi\colon S^n \times S^1 \to S^n$, $n \geqslant 2$, satisfies all the hypotheses of Theorem 2.4 except the semisimplicity condition. $S^n \times S^1$ does not admit a metric with Ric > 0 since it has infinite fundamental group.

3. Positive Ricci curvature on compact bundles

We begin this section by constructing a family of metrics $\langle\ ,\ \rangle_{E,t}$ on the total space E of a bundle, and then derive estimates for the Ricci curvature of these metrics. These estimates will be used to obtain positive Ricci curvature on several classes of bundles.

For M^n compact, consider a bundle $\hat{\pi}\colon E \to M$ with fibre F^r, structural group G^m, and associated principal bundle $\pi\colon P \to M$. Assume that F admits a G-invariant metric $\langle\ ,\ \rangle_F$. For a fixed connection ω on P, bi-invariant metric $\langle\ ,\ \rangle_G$ on G, and metric $\langle\ ,\ \rangle_M$ on M, we obtain a family of metrics on $P \times F$: $\langle\ ,\ \rangle_t^{\sim} = \langle\ ,\ \rangle_t \times \langle\ ,\ \rangle_{F,t}$, where $\langle\ ,\ \rangle_{F,t} = t^2\langle\ ,\ \rangle_F$. These metrics are G-invariant and thus induce, by horizontal projection, a family of metrics $\langle\ ,\ \rangle_{E,t}$ on $E = (P \times F)/G$ such that $\tilde{\pi}\colon P \times F \to (P \times F)/G$ is a Riemannian submersion for each t.

For $(p, y) \in P \times F$ let $\tilde{V}_{p,y}$ and $\tilde{H}_{p,y}$ denote the vertical and horizontal subspaces respectively at (p, y). For an orthonoraml basis $e_1, \ldots, e_m$ of $\mathfrak{g}$ let $\bar{e}_1, \ldots, \bar{e}_m$ and $\tilde{e}_1, \ldots, \tilde{e}_m$ be the associated fields on P and F respectively. Since $\{\bar{e}_i(p) - \tilde{e}_i(y),\ i = 1, \ldots, m\}$ is a basis of $\tilde{V}_{p,y}$, both $\tilde{H}_{p,y}$ and $\tilde{V}_{p,y}$ are independent of t.

Proposition 3.1. *Let $\hat{\pi}\colon E \to M$ be a fibre bundle, and $\langle\ ,\ \rangle_F$ a complete G-invariant metric on F. Then $\langle\ ,\ \rangle_{E,t}$ is complete, and the fibres are totally geodesic and mutually isometric.*

Proof. Set $\pi_1(p, y) = \pi(p)$. Then the diagram

$$\begin{array}{ccc} P \times F & & \\ \tilde{\pi} \downarrow & \searrow \pi_1 & \\ E & \underset{\hat{\pi}}{\rightarrow} & M \end{array} \tag{3.1}$$

commutes. Fix $\xi = \tilde{\pi}(p, y) \in E$, and let $u \in T_\xi E$. Let $\tilde{u} = (v, w) \in \tilde{H}_{p,y}$ be the horizontal lift of u with $v \in T_p P$ and $w \in T_y F$. If $\tilde{c}$: $\mathbf{R} \to P \times F$, $\tilde{c}(s) = (\alpha(s), \beta(s))$, is the unique geodesic with $\dot{\tilde{c}}(0) = \tilde{u}$, then by O'Neill [11] $c = \tilde{\pi} \circ \tilde{c}$ is a geodesic with $\dot{c}(0) = u$. Hence $\langle\ ,\ \rangle_{E,t}$ is complete. Assume further that u is tangent to F_x, the fibre through x, $x = \hat{\pi}(\xi)$. By (3.1), $\pi_1(\tilde{u}) = 0$ which implies $v \in V_p$. Because the fibres in P are totally geodesic, $\pi(\dot{\alpha}(s)) \equiv 0$. Again by (3.1), $\hat{\pi}(\dot{c}(s)) \equiv 0$. Thus F_x is totally geodesic.

Define π_p: $F \to E$ by $\pi_p(y) = \tilde{\pi}(p, y)$. To prove the fibres are mutually isometric it suffices to show that for fixed t the pull-back metrics $\pi_p^*(\langle\ ,\ \rangle_{E,t})$ are independent of p. For $u \in T_y F$ its length in the metric $\pi_p^*(\langle\ ,\ \rangle_{E,t})$ is given by $\|v_p + w_p\|_t^{\sim}$ where $v_p \in T_p P$, $w_p \in T_y F$, and $v_p + w_p$ is the unique vector in $\tilde{H}_{p,y}$ such that $v_p + w_p - u \in \tilde{V}_{p,y}$. Fix $q \in P$. There exists $e \in \mathfrak{g}$ such that $\bar{e}(q) - \bar{e}(y) = v_q + w_q - u$, and thus $v_q + w_q = \bar{e}(q) + u - \bar{e}(y)$. Setting $z_p = \bar{e}(p) + u - \bar{e}(y)$ one notes that z_p is horizontal (since z_q is) and $z_p - u$ is vertical. Thus $z_p = v_p + w_p$, and since $\|z_p\|_t^{\sim} = \|z_q\|_t^{\sim}$, the norm is independent of p.

The lemma below will be useful in later calculations.

Lemma 3.2. *Let $v_1, \ldots, v_n$ be an orthonormal set of vectors in an inner product space V. Suppose that $(\alpha_{ij}) \in O(n)$ and that $c_1, \ldots, c_n$ are constants such that $c_i \geqslant \rho > 0$, $i = 1, \ldots, n$. Set $y_i = \sum_j \alpha_{ij} c_j v_j$. Then*

$$\|y_i \wedge y_j\|^2 = \|y_i\|^2 \|y_j\|^2 - \langle y_i, y_j \rangle^2 \geqslant \rho^4 \text{ for } i \neq j.$$

Proof.

$$\begin{aligned} \|y_i\|^2 \|y_j\|^2 - \langle y_i, y_j \rangle^2 &= \Big(\sum_k \alpha_{ik}^2 c_k^2\Big)\Big(\sum_l \alpha_{jl}^2 c_l^2\Big) - \Big(\sum_k \alpha_{ik}\alpha_{jk} c_k^2\Big)\Big(\sum_l \alpha_{il}\alpha_{jl} c_l^2\Big) \\ &= \sum_{k,l} \big(\alpha_{ik}^2 \alpha_{jl}^2 - \alpha_{ik}\alpha_{jk}\alpha_{il}\alpha_{jl}\big) c_k^2 c_l^2 \\ &= \tfrac{1}{2} \sum_{k,l} (\alpha_{ik}\alpha_{jl} - \alpha_{il}\alpha_{jk})^2 c_k^2 c_l^2 \\ &\geqslant \tfrac{1}{2} \sum_{k,l} (\alpha_{ik}\alpha_{jl} - \alpha_{il}\alpha_{jk})^2 \rho^4 = \rho^4. \end{aligned}$$

For a bundle $\hat{\pi}$: $E \to M$ with the conditions sufficient to define $\langle\ ,\ \rangle_{E,t}$ we wish to obtain an estimate of the Ricci curvature. Fix $\overline{X} \in T_\xi E$, $\|\overline{X}\|_{E,t} = 1$, where t is also fixed. There exists $X \in \tilde{H}_{p,y}$ with $\tilde{\pi}(X) = \bar{X}$. Choose orthonormal bases (with respect to the 1-metric) $e_1, \ldots, e_m$ and $f_1, \ldots, f_s$, $k_1, \ldots, k_{r-s}$ of $\mathfrak{g}$ and $T_y F$ respectively such that

$$\begin{aligned} \tilde{e}_l(y) &= \lambda_l(y) f_l, \quad l = 1, \ldots, s \qquad (\lambda_l(y) > 0), \\ &= 0, \qquad\quad l = s+1, \ldots, m. \end{aligned}$$

The vectors $v_1 - \lambda_1(y) f_1, \ldots, v_s - \lambda_s(y) f_s, v_{s+1}, \ldots, v_m$ form a basis for $\tilde{V}_{p,y}$, where $v_i = \bar{e}_i(p)$. If $h_1, \ldots, h_n$ is an orthonormal basis for $H_p \subset T_p P$, then $h_1, \ldots, h_n, \hat{k}_1, \ldots, \hat{k}_{r-s}, w_1, \ldots, w_s$ is an orthonormal basis for $\tilde{H}_{p,y}$ with respect to $\langle\, , \rangle_t^{\sim}$, where

$$\hat{k}_j = (t^{-1}) k_j, \qquad j = 1, \ldots, r-s,$$

$$w_l = \left[t(\lambda_l^2(y) + 1)^{1/2} \right]^{-1} (\lambda_l(y) v_l + f_l), \; l = 1, \ldots, s.$$

Let H, K, J denote the spaces spanned by the h_i, k_j, and w_l respectively. There exist $h \in H$, $\hat{k} \in K$, and $z_1 \in J$ all unit vectors for $\| \, \|_t^{\sim}$ such that $X = a(\alpha z_1 + \beta \hat{k}) + bh$ with $a^2 + b^2 = \alpha^2 + \beta^2 = 1$. By choosing new bases for H and K we may assume $h = h_1$ and $\hat{k} = \hat{k}_1$. Complete z_1 to an orthonormal basis $z_1, \ldots, z_s$ of J. There exists $(\alpha_{lj}) \in O(s)$ such that $z_l = \sum_{j=1}^s \alpha_{lj} w_j$. Set

$$u_l = \sum_j \left[t(\lambda_j^2(y) = 1)^{1/2} \right]^{-1} (\lambda_j(y) \alpha_{lj} v_j),$$

$$\hat{u}_l = \sum_j \left[t(\lambda_j^2(y) + 1)^{1/2} \right]^{-1} \alpha_{lj} f_j, \; l = 1, \ldots, s,$$

so that $z_l = u_l + \hat{u}_l$. We have the following orthonormal basis for $\tilde{H}_{p,y}$:

$$\begin{aligned} X &= a(\alpha z_1 + \beta \hat{k}_1) + b h_1, \\ Y &= b(\alpha z_1 + \beta \hat{k}_1) - a h_1, \\ Z &= \beta z_1 - \alpha \hat{k}_1, \end{aligned}$$

$$\hat{k}_2, \ldots, \hat{k}_{r-s}, h_2, \ldots, h_n, z_2, \ldots, z_l.$$

By (1.3) the sectional curvature of a plane σ spanned by horizontal vectors on $P \times F$ is increased under projection by $\tilde{\pi}$. Thus

$$\begin{aligned} \mathrm{Ric}_{E,t}(\bar{X}) \geqslant \tilde{R}_t(X, Y) &+ \tilde{R}_t(X, Z) + \sum_{i=2}^{n} \tilde{R}_t(X, h_i) \\ &+ \sum_{j=2}^{r-s} \tilde{R}_t(X, \hat{k}_j) + \sum_{l=2}^{s} \tilde{R}_t(X, z_l). \end{aligned} \tag{3.2}$$

In the estimates below any C_i is a positive constant depending on ω, $\langle\, , \rangle_M$, and $\langle\ ,\ \rangle_G$. The following inequalities are obtained by using (2.2):

$$\begin{aligned}
\tilde{R}_t(X, Y) &= \alpha^2 R_t(u_1, h_1) \geqslant 0,\\
\tilde{R}_t(X, Z) &= a^2 R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \beta\hat{u}_1 - \alpha\hat{k}_1\big) + R_t(a\alpha u_1 + bh_1, \beta u_1)\\
&\geqslant a^2 R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \beta\hat{u}_1 - \alpha\hat{k}_1\big)\\
\tilde{R}_t\big(X, \hat{k}_j\big) &= a^2 R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \hat{k}_j\big),\\
\tilde{R}_t(X, z_l) &\geqslant a^2 R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \hat{u}_l\big) + t^2 a^2 \alpha^2 R(u_1, u_l),
\end{aligned}$$

$$|R_t(u_1, h_i, h_i, h_1)| \leqslant t \sum_k |\alpha_{1k} R(v_k, h_i, h_i, h_1)| \leqslant tC_1,$$

and thus

$$\begin{aligned}
\tilde{R}_t(X, h_i) = R_t(a\alpha u_1 + bh_1, h_i) &\geqslant b^2 R_t(h_1, h_i) + 2ab\alpha R_t(u_1, h_i, h_i, h_1)\\
&\geqslant b^2 R_M(\pi(h_1), \pi(h_i)) - t^2 b^2 C_2 - t|ab\alpha| C_3.
\end{aligned}$$

These estimates and (3.2) yield

$$\begin{aligned}
\operatorname{Ric}_{E,t}(\overline{X}) \geqslant{}& b^2 \operatorname{Ric}_M(\pi(h_1)) + (ta\alpha)^2 \sum_{l=2}^{s} R(u_1, u_l)\\
&+ a^2\Bigg[R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \beta\hat{u}_1 - a\hat{k}_1\big) + \sum_{j=2}^{r-s} R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \hat{k}_j\big)\\
&+ \sum_{l=2}^{s} R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \hat{u}_l\big)\Bigg] - (tb)^2 C_2 - t|ab\alpha| C_3.
\end{aligned} \tag{3.3}$$

Define $0 < \rho(y) = \min_i\{(\lambda_i^2(y) + 1)^{-1/2}\} \leqslant 1$. Because $f_1/t, \ldots, f_s/t$, $\hat{k}_1, \ldots, \hat{k}_{r-s}$ are orthonormal, Lemma 3.2 can be applied to $\alpha\hat{u}_1 + \beta\hat{k}_1$, $\beta\hat{u}_1 - \alpha\hat{k}_1$, $\hat{u}_2, \ldots, \hat{u}_s$, $\hat{k}_2, \ldots, \hat{k}_{r-s}$. Thus with respect to $\|\ \|_{F,t}$ the plane spanned by any two of the above vectors has $\rho^4(y)$ as a lower bound for its norm squared. Let K_F denote the sectional curvature of $\langle\ ,\ \rangle_F$. Using the above and $R(u_1, u_l) \geqslant 0$ we simplify (3.3) to

$$\begin{aligned}
\operatorname{Ric}_{E,t}(\overline{X}) \geqslant{}& b^2 \operatorname{Ric}_M(\pi(h_1)) + (\rho^4(y) a^2/t^2)\Bigg[K_F(\hat{u}_1, \hat{k}_1)\\
&+ \sum_j K_F\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \hat{k}_j\big) + \sum_l K_F\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \hat{u}_l\big)\Bigg]\\
&- (tb)^2 C_2 - t|ab\alpha| C_3.
\end{aligned} \tag{3.4}$$

Definition. (F, G) satisfies *Condition A* if G is a compact Lie group acting by isometries on a compact Riemannian manifold F with positive Ricci curvature such that each λ_i: $F \to \mathbf{R}$ is either identically 1 or 0. If (F, G)

satisfies Condition A, the dimensions of the orbits of F are identically, in fact, equal to the number of λ_i with constant value 1.

Theorem 3.3. *Let (E, M, F, G) be a fibre bundle with M compact and admitting a metric such that $\mathrm{Ric}_M \geqslant M_0 > 0$. If (F, G) satisfies Condition A, E admits a metric of positive Ricci curvature.*

Proof. In the previously developed notation $\hat{u}_i = (t\sqrt{2}\,)^{-1}\Sigma_j\, \alpha_{ij} f_j$ with the Condition A assumption. The plane $\sigma = \{\hat{u}_1, \hat{k}_1\}$ is orthogonal to $\hat{u}_2, \ldots, \hat{u}_s$, $\hat{k}_2, \ldots, \hat{k}_{r-s}$. Thus (3.4) becomes

$$\mathrm{Ric}_{E,t}(\overline{X}) \geqslant b^2 M_0 + (a^2/4t^2)\, \mathrm{Ric}_F(\alpha\hat{u}_1 + \beta\hat{k}_1) + 0(t).$$

Since there exists a constant F_0 satisfying $\mathrm{Ric}_F \geqslant F_0 > 0$, the theorem is proved by choosing t small enough.

We now find fibres F admitting actions which satisfy Condition A. If F admits a transitive action by a compact Lie group G, then $F = G/H$ for $H \subset G$ a closed subgroup, and the action is by left multiplication.

Proposition 3.4. *For a compact Lie group G and closed subgroup H the following are equivalent:*

(i) *$\pi_1(G/H)$ is finite.*

(ii) *Any normal homogeneous metric on G/H has positive Ricci curvature.*

(iii) *G/H admits a metric of positive Ricci curvature.*

Proof. (ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (i) is standard. To prove (i) $\Rightarrow$ (ii) we first exhibit a compact semi-simple group $\tilde{G}$ which acts transitively on G/H. The Lie algebra $\mathfrak{g}$ of G can be decomposed as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{z}$ where $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}]$ is semi-simple and $\mathfrak{z}$ is the center of $\mathfrak{g}$. There exists a covering $\pi\colon \tilde{G} \times T^n \to G$ such that $\tilde{G}$ is compact, simply connnected, semi-simple, and has Lie algebra $\mathfrak{k}$, and T^n is a torus with algebra $\mathfrak{z}$. Let $\tilde{H}$ denote the path component of the identity of $\pi^{-1}(H)$. Since the induced mapping $\bar{\pi}\colon (\tilde{G} \times T)/\tilde{H} \to G/H$ is a covering, $\pi_1[(\tilde{G} \times T)/\tilde{H}]$ is also finite. It suffices to show that $\tilde{G}$ acts transitively on $(\tilde{G} \times T)/\tilde{H}$ to conclude that it acts transitively on G/H. Let $i\colon \tilde{H} \to \tilde{G} \times T$ be the natural injection, and $p\colon \tilde{G} \times T \to T$ denote projection to the second factor. The problem reduces to showing $p \circ i$ is surjective. If not, $T/\overline{H}$ is a torus for $\overline{H} = p \circ i(\tilde{H})$, and thus $\pi_1(T/\overline{H})$ is infinite. Let $\alpha \in \pi_1(T/\overline{H})$ and $q\colon T \to T/\overline{H}$ denote projection. By the long exact homotopy sequence for $\overline{H} \to T \to T/\overline{H}$ there exists $\beta \in \pi_1(T)$ such that $q_*\beta = \alpha$. Let $j\colon T \to (\tilde{G} \times T)/\tilde{H}$ denote inclusion followed by projection, and $\bar{p}$: $(\tilde{G} \times T)/\tilde{H} \to T/H$ denote the map induced by p. Since $q = \bar{p} \circ j$, $\bar{p}_*(j_*\beta) = \alpha$. Thus $\bar{p}_*$ is surjective contradicting the fact that $\pi_1[(\tilde{G} \times T)/\tilde{H}]$ is finite.

We recall the following formulas for a normal homogeneous metric induced from $\langle\ ,\ \rangle_G$ on G, [8]:

(3.5) $$\langle [Z, X], Y\rangle_G = \langle X, [Y, Z]\rangle_G, X, Y, Z \in \mathfrak{g},$$

(3.6) $$R(X, Y) = \tfrac{1}{4}\|[X, Y]_{\mathfrak{M}}\|_G^2 + \|[X, Y]_{\mathfrak{h}}\|_G^2, X, Y \in \mathfrak{M},$$

where R is the curvature tensor associated with the metric on G/H. In (3.6), $\mathfrak{M} = \mathfrak{h}^{\perp}$, the orthogonal complement of $\mathfrak{h}$, the Lie algebra $\mathfrak{h}$ of H. We make the usual identification $\mathfrak{M} \simeq T_{[H]}G/H$. From (3.6), $\mathrm{Ric} \geqslant 0$ on G/H. Suppose $X \in \mathfrak{M}$, $X \neq 0$ satisfies $\mathrm{Ric}(X) = 0$. Then (3.6) implies $[X, Y] = 0$ for all $Y \in \mathfrak{M}$. Fix $Z \in \mathfrak{h}$. For arbitrary $Y \in \mathfrak{M}$, $\langle [X, Z], Y\rangle_G = 0$ by (3.5), and $[X, Z] = 0$ since $[X, Z] \in \mathfrak{M}$. Thus $X \in \mathfrak{z}$. Since $\tilde{G}$ acts transitively on G/H, the map $\mathfrak{k} \hookrightarrow \mathfrak{M} + \mathfrak{h} \to \mathfrak{M}$ is surjective. Thus there exist $A \in \mathfrak{k}$ and $B \in \mathfrak{h}$ such that $A = X + B$. By (3.5) and the fact that $\mathfrak{k} = [\mathfrak{g}, \mathfrak{g}]$, X is orthogonal to A. But since also $X \in h^{\perp}$, $X = 0$.

Theorem 3.5. *Let $(E, M, G/H, G, \tilde{\pi})$ be a fibre bundle such that M is compact and admits a metric with* $\mathrm{Ric}_M \geqslant 0$. *If G/H admits a metric with* $\mathrm{Ric} > 0$ *(equivalently, if $\pi_1(G/H)$ is finite), then E admits a metric of positive Ricci curvature.*

Proof. By Theorem 3.3 it remains to show that $(G/H, G)$ satisfies Condition A for G/H with a normal homogeneous metric. By Proposition 3.4 this metric has positive Ricci curvature. Since a normal homogeneous metric is G-invariant, the λ_i are identically 1 or 0.

Remark. In the above theorem the action of G on G/H is assumed to be the standard one. The following corollary is immediate.

Corollary 3.6. *Let $(E, M, S^n, O(n+1), \tilde{\pi})$, $n \geqslant 2$ be a sphere bundle whose base is compact admitting a metric with* $\mathrm{Ric}_M > 0$. *Then E admits a metric with* $\mathrm{Ric} > 0$.

Eells and Kuiper [2] have shown that a number of exotic 7- and 15-spheres arise as sphere bundles over a standard spheres. These exotic spheres will thus admit metrics of positive Ricci curvature. More precisely, we have the following.

Theorem 3.7. *Of the* 28 (16, 256) *diffeomorphisms classes of* 7- (15-) *spheres,* 16 (4,096) *admit metrics of positive Ricci curvature.*

Not all exotic spheres admit such metrics. In fact, Hitchen [6] has proved that any exotic sphere which does not bound a spin manifold cannot even admit a metric of positive scalar curvature. For $n = 1$ or 2 (mod 8) the exotic spheres which bound spin manifolds form a subgroup of index 2 in the group Θ_n of homotopy n-spheres. Since all 7- and 15-spheres bound spin manifolds, there are no known obstructions to metrics of positive Ricci curvature on any of them. Hernandez [5] has constructed metrics of positive Ricci curvature on a large number of exotic spheres which occur as certain Brieskorn varieties. His results do not seem to include the spheres in the theorem above.

We now consider another class of examples: Fibres F admitting a free action by a compact semi-simple group G. If F/G admits a metric with $\mathrm{Ric} > 0$, then Theorem 2.4 implies that the metric $t^{-2}\langle\ ,\ \rangle_t$ on F has positive Ricci curvature for t small enough. By construction it satisfies Condition A. This yields the following result.

Theorem 3.8. *Let $(E, M, F, G, \hat{\pi})$ be a bundle with M compact and admitting a metric with $\mathrm{Ric}_M > 0$. If G is compact semisimple and acts freely on F which is compact such that F/G admits a metric with $\mathrm{Ric} > 0$, then E admits a metric of positive Ricci curvature.*

We conclude this section by noting that hypotheses on the type of action on F can be reduced if the curvature assumption on F is strengthened. The following result was obtained independently by Poor [12] who used it to prove Theorem 3.7.

Theorem 3.9. *Let $(E, M, F, G, \hat{\pi})$ be a bundle with M compact and admitting a metric with $\mathrm{Ric}_M > 0$, and suppose G is compact. If F is compact and admits a G-invariant metric of positive sectional curvature, then B admits a metric of positive Ricci curvature.*

Proof. This follows immediately from (3.4) by choosing t small enough.

Remark. By Proposition 3.1 the fibres are totally geodesic and mutually isometric with respect to the metrics on E in the theorems in this section.

4. Positive Ricci curvature on vector bundles

In this section G will be the m-dimensonal orthogonal group $\mathrm{O}(m)$ whose Lie algebra $\mathfrak{g} = \mathfrak{o}(m)$ consists of the $m \times m$ skew-symmetric matrices. Let E_{ij} denote the $m \times m$ matrix whose (i,j) entry is 1 and all otheres are 0. The matrix $A_{ij} = E_{ij} - E_{ji}$ is skew-symmetric, and $\{A_{ij}\}_{i<j}$ form a basis for $\mathfrak{o}(m)$. The fact that $E_{ij} \cdot E_{kl} = \delta_{jk} E_{il}$ implies

$$[A_{ij}, A_{kl}] = \delta_{jk} A_{il} + \delta_{lj} A_{ki} + \delta_{ki} A_{lj} + \delta_{il} A_{jk}. \tag{4.1}$$

The inner product

$$\langle \alpha, \beta \rangle = -(1/2)\,\mathrm{trace}(\alpha\beta),\ \alpha, \beta \in \mathfrak{g}$$

on $\mathfrak{g}$ defines a metric $\langle\ ,\ \rangle_G$ on G by left translation which is bi-invariant. The matrices $\{A_{ij}\}_{i<j}$ form an orthonormal basis with respect to $\langle\ ,\ \rangle_G$. This metric will remain fixed throughout this section.

Let $v \cdot w$ denote the usual inner product of two vectors in $\mathbf{R}^m$, and $\varepsilon_1, \dots, \varepsilon_m$ be the canonical orthogonormal basis with respect to this inner product. For $x \in \mathbf{R}^m$ we make the standard identification $T_x\mathbf{R}^m \simeq \mathbf{R}^m$.

The m-dimensional paraboloid F is diffeomorphic to $\mathbf{R}^m$ and has positive sectional curvature. F is obtained from the induced metric on $\mathbf{R}^m$ via the imbedding $i: \mathbf{R}^m \to \mathbf{R}^{m+1}$, $i(x) = (x, x \cdot x/2)$. For $v, w \in T_xF$, $i_*(v) = (v, v \cdot x)$ so that $\langle v, w\rangle_F = v \cdot w + (v \cdot x)(w \cdot x)$. Thus F is $\mathrm{O}(m)$-invariant. The curvature of F is computed by noting that its image in $\mathbf{R}^{m+1}$ is the 0-set for the function f,

$$f(x_1, \ldots, x_{m+1}) = x_1^2 + \cdots + x_m^2 - 2x_{m+1}.$$

From $\nabla f(y) = 2(y_1, \ldots, y_m, -1)$ it follows [3, p. 109] that for $u, v, w, z, \in T_xF$

$$R_F(u, v, w, z) = (|x|^2 + 1)^{-1}\big[(v \cdot w)(u \cdot z) - (u \cdot w)(v \cdot z)\big]. \tag{4.2}$$

Thus for $m \geqslant 2$, F has positive sectional curvature, and is complete since its image in $\mathbf{R}^{m+1}$ is closed.

The action of $\mathrm{O}(m)$ at $y = r\varepsilon_m \in F$ is especially simple. Since the action is linear $\bar{A}(y) = Ay$, $A \in \mathfrak{o}(m)$. Thus

$$\begin{aligned} \bar{A}_{im}(y) &= r\varepsilon_i, \quad && i = 1, \ldots, m-1, \\ \bar{A}_{ij}(y) &= 0, && i < j < m. \end{aligned} \tag{4.3}$$

Theorem 4.1. *Let $(E, M, \mathbf{R}^m, \mathrm{O}(m), \hat{\pi})$ be a vector bundle with $m > 2$, and assume M is compact and admits a metric with $\mathrm{Ric}_M > 0$. Then E admits a complete metric of positive Ricci curvature such that the fibres are totally geodesic and mutually isometric.*

Proof. We use the metric $\langle\ ,\ \rangle_G$ and $\langle\ ,\ \rangle_G$ introduced above to define $\langle\ ,\ \rangle_{E,t}$ as in Proposition 3.1. It remains to show that $\mathrm{Ric}_{E,t} > 0$ for t small enough.

For $\xi \in E$ there exists $(p, y) \in P \times F$ such that $\tilde{\pi}(p, y) = \xi$ with $y = r\varepsilon_m$, $r \geqslant 0$. Thus in the calculations leading to (3.3) we assume y is of this form. From (4.3) we have $\lambda_i(y) = r$, $i = 1, \ldots, m-1$. Note that $\varepsilon_1, \ldots, \varepsilon_{m-1}$, $\varepsilon_m/(r^2 + 1)^{1/2}$ is an orthonormal basis for the tangent space at $r\varepsilon_m$. Thus (3.3) holds with the following substitutions:

$$u_l = r\big[t(r^2 + 1)^{1/2}\big]^{-1} \cdot \sum_{j=1}^{m-1} \alpha_{lj} A_{jm}, \quad l = 1, \ldots, m-1,$$

$$\hat{u}_l = \big[t(r^2 + 1)^{1/2}\big]^{-1} \cdot \sum_{j=1}^{m-1} \alpha_{lj} \varepsilon_j, \quad l = 1, \ldots, m-1,$$

$$\hat{k}_1 = \big[t(r^2 + 1)^{1/2}\big]^{-1} \cdot \varepsilon_m.$$

In (3.3) the terms in brackets involving the curvature of F become

$$R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \beta\hat{u}_1 - \alpha\hat{k}_1\big) + \sum_{l=2}^{m-1} R_{F,t}\big(\alpha\hat{u}_1 + \beta\hat{k}_1, \hat{u}_l\big)$$
$$= (m-1)\big[t^2(r^2+1)^3\big]^{-1},$$

because of (4.2). By (4.1),

$$\sum_{l=1}^{m-1} R(u_1, u_l) = r^4\big[4(r^2+1)^2t^4\big]^{-1} \cdot \sum_{j,k,l} \|\big[\alpha_{1j}A_{jm}, \alpha_{lk}A_{km}\big]\|_G^2$$
$$= r^4\big[4(r^2+1)^2t^4\big]^{-1} \cdot \sum_{j,k,l} \alpha_{1j}^2\alpha_{lk}^2(1-\delta_{jk})$$
$$= (m-2)r^4\big[4(r^2+1)^2t^4\big]^{-1}.$$

Referring to (3.3) the assumption on M implies that there exists $M_0 > 0$ such that $\mathrm{Ric}_M(\pi(h_1)) - (tb)^2C_2 > M_0$ for t small enough.

Combining all the above with (3.3) produces

$$\mathrm{Ric}_{E,t}(\overline{X}) \geqslant b^2M_0 - t|ab\alpha|C_3 + \left(\frac{a\alpha}{t}\right)^2 f(r) + \left(\frac{a}{t}\right)^2 g(r)$$
$$\geqslant \tfrac{1}{2}\left[b^2M_0 + \left(\frac{a}{t}\right)^2 g(r)\right] \tag{4.4}$$
$$+ \left\{\tfrac{1}{2}b^2M_0 - |ab\alpha|tC_3 + \left(\frac{a\alpha}{t}\right)^2\big[f(r) + \tfrac{1}{2}g(r)\big]\right\}$$

for t small enough, where $f(r) = (m-2)r^4[2(r^2+1)]^{-2}$ and $g(r) = (m-1)[r^2+1]^{-3}$. Denote the term in brackets { } by Λ. The function $f(r) + \frac{1}{2}g(r)$ is bounded away from zero for $r \geqslant 0$. Thus there exist positive constants C_4, C_5, and C_6 such that

$$\Lambda \geqslant C_4b^2 - |ab\alpha|tC_5 + (a\alpha/t)^2C_6$$
$$\geqslant (a\alpha)^2\big[C_6t^{-2} - (C_5t)^2(4C_4)^{-1}\big] \geqslant 0 \tag{4.5}$$

for t small enough. The second inequality in (4.5) follows since the right-hand term is quadratic in $|b|$ so that its minimum is easily computed. For t small enough, (4.4) and (4.5) imply

$$\mathrm{Ric}_{E,t}(\overline{X}) \geqslant \tfrac{1}{2}\big[b^2M_0 + (a/t)^2g(r)\big] > 0.$$

In [4] Gromoll and Meyer show that for M compact, $M \times \mathbf{R}$ does not admit a complete metric of positive Ricci curvature. If M is compact with $\mathrm{Ric}_M > 0$, we know of no obstruction to constructing metrics of positive Ricci curvature on vector bundles with fibre dimension equal to two.

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